CHAPTER 1- FUNCTIONS

1.1 RELATIONS

1.1.1 Representing Relations

1- By Arrow Diagram

One to one relation

\[ \begin{array}{c}
1 & \rightarrow & 3 \\
2 & \rightarrow & 4 \\
3 & \rightarrow & 5
\end{array} \]

One to many relation

\[ \begin{array}{c}
2 & \rightarrow & 4 \\
-2 & \rightarrow & 4 \\
3 & \rightarrow & 9 \\
-3 & \rightarrow & -3
\end{array} \]

Many to one relation

\[ \begin{array}{c}
1 & \rightarrow & 2 \\
2 & \rightarrow & 4 \\
3 & \rightarrow & 9 \\
-2 & \rightarrow & 4 \\
3 & \rightarrow & 9 \\
-3 & \rightarrow & 9
\end{array} \]

Many to many relation

\[ \begin{array}{c}
6 \text{ hours} & \rightarrow & 360 \text{ min} \\
1 \text{ min} & \rightarrow & 60 \text{ sec} \\
2160 \text{ sec} & \rightarrow & 1 \text{ hour} \\
1 \text{ hour} & \rightarrow & 3600 \text{ sec} \\
0.25 \text{ day} & \rightarrow & 360 \text{ min} \\
60 \text{ sec} & \rightarrow & 1 \text{ min} \\
0.25 \text{ day} & \rightarrow & 2160 \text{ sec}
\end{array} \]

2- By Ordered Pair

\[(1, 3), (2, 4), (3, 5)\]

\[(1, 4), (-2, 4), (3, 9), (-3, 9)\]

3. By Graph

\[y\text{ (Image)}\]

\[\begin{array}{c}
4 \\
3 \\
2 \\
1 \\
0
\end{array}\]

\[\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}\]

x (object)
Set A is called **domain** of the relation and set B is the **codomain**.
Each of the elements in the **domain** (Set A) is an object. So 1, 2 and 3 are called **object**.
Each of the elements in the **codomain** (Set B) is an image. So 3, 4, 5 and 6 are called **image**.
The set that **contains all the images** that are matched is the **range**.

**1.2 FUNCTIONS**

1. Function is a **special relation** where **every object** in a domain **has only one image**. A function is also known as a mapping.

2. From the types of relation we have learned, only **one to one relation** and **many to one relation** are function.

**Function notation**

Small letter: f, g, h or something else...

\[ f : x \rightarrow 2x \]

Read as function f map x onto 2x

**Example:**

\[ f : x \rightarrow 2x \]

\[ f(x) = 2x \]

\[ x = 1, \]

\[ f(1) = 2(1) = 2 \]

Look at the difference between these two.
Example 1:

1. Given function \( f : x \rightarrow 3x - 2 \). Find
   (a) the image of \(-1\)
   
   \[
   f(x) = 3x - 2 \\
   f(-1) = 3(-1) - 2 \\
   = -3 - 2 \\
   = -5
   \]

   (b) object which has the image 4

   \[
   f(x) = 3x - 2 \\
   \text{Given that the image is 4,} \\
   \text{So } f(x) = 4
   \]

   Compare 1 and 2, 
   Hence, 
   \[
   3x - 2 = 4 \\
   3x = 6 \\
   x = 2
   \]

2. Given that \( f(x) = px + 3 \) and \( f(4) = 5 \). Find the value of \( p \).

   \[
   f(x) = px + 3 \\
   \text{If } x=4, \\
   f(4) = 4p + 3 \\
   f(4) = 5
   \]

   Compare 1 and 2, 
   Hence, 
   \[
   4p + 3 = 5 \\
   4p = 2 \\
   p = \frac{1}{2}
   \]
Example 2:

The function $g$ is defined by $g(x) = \frac{2x}{x + m}$ if $g(5) = 3g(2)$, find the value of $m$. Hence, find:

(a) image of 4.
(b) the value of $x$ such that the function $g$ is undefined.

Answer:

$$g(5) = 3g(2)$$
$$\frac{2(5)}{5 + m} = 3 \cdot \frac{2(2)}{2 + m}$$
$$60 + 12m = 20 + 10m$$
$$2m = -40$$
$$m = -20$$

(a) $g(4) = \frac{2(4)}{4 - 20}$
$$= -\frac{1}{2}$$

(b) $x - 20 = 0$
$$x = 20$$

Any number that is divided by zero will result undefined or infinity. The value of denominator cannot equal to zero because it would cause the solution becomes undefined or infinity. To find the value of $x$ to make the function undefined, the denominator must be equal to zero.

EXERCISE 1.2

1. Given function $f$ is defined by $f(x) = 5x - 4$. Find the value of $x$ if $f(x) = -14$.

2. Given that $f(x) = 2x - 3$. Find the image of 5.

3. Given that function $f$ such that $f(x) = \frac{2x - 3}{x - 4}$.

   (i) The image of 5.
   (ii) the value of $x$ such that the function $f$ is undefined

4. Given the function $h$ such that $h(x) = 2x^2 + 3kx + 1$ and $h(1) = 9$. Find the value of $k$.

5. Given the function $f : x \rightarrow 3x + 1$, find

   (a) the image of 2,
   (b) the value of $x$ if $f(x) = x$.

6. Given the function $w : x \rightarrow 4x + 3$, find the value of $p$ such that $w(p) = 23$.

7. Given the function $n : x \rightarrow 2x - 5$. Find the value of $k$ at which $n(k) = k$. 
1.3 COMPOSITE FUNCTIONS

The figure shows Set A, B and C. The combined effect of two functions can be represented by the function gf(x) and not fg(x). This is because the range of f has become the domain of function g or f(x) becomes the domain of function g.

Example 1:
Given function \( f : x \to x + 1 \) and \( g : x \to 3x \). Find the composite function \( gf \).

\[
\begin{align*}
\text{Replace } f(x) \text{ with its image which is } (x +1) \\
\text{(x +1) has become the object of function g which is replacing x. [g(x)=3x and g(x+1)=3(x+1)]. When x in the object is replaced by (x+1) so x in the image is also replaced by (x+1) and become 3(x+1). We can also expand the expression and becomes 3x + 3.}
\end{align*}
\]

Hence, \( gf : x \to 3(x +1) \)

Example 2:
The following information refers to the functions \( f \) and \( g \).

\[
\begin{align*}
f &: x \to 2x - 3 \\
g &: x \to \frac{4}{x - 1}, \quad x \neq k
\end{align*}
\]

Find
(a) the value of \( k \),
(b) \( fg(x) \).
Solution:
In Form One we have learned that any number that divided by zero will result infinity or undefined.
For part (a), if there is given an expression which is a fraction there must be given an information that
the value of unknown in the denominator that it cannot be. For example, \( g : x \rightarrow \frac{h}{x - 2}, \quad x \neq 2 \). The
part of denominator is \( x - 2 \). If \( x = 2 \), the denominator would be 0 and the solution will be undefined
or infinity. So \( x \) cannot equal to 2. For the question above, we know that \( x \) cannot equal to 1 because if
\( x = 1 \), the denominator would be 0 and the solution will be undefined or infinity. It is given that \( x \neq k \).
We already know that \( x \neq 1 \). So compare these two equations and it would result \( k = 1 \). So 1 is the
answer for part (a).

For part (b),
\[
f(x) = 2x - 3 \text{ and } \quad g(x) = \frac{4}{x - 1}, \quad x \neq 1
\]
\[
fg(x) = f[g(x)]
\]
\[
= f\left(\frac{4}{x - 1}\right)
\]
\[
= 2\left(\frac{4}{x - 1}\right) - 3
\]
\[
= \frac{8}{x - 1} - 3
\]
\[
= \frac{8 - 3(x - 1)}{x - 1}
\]
\[
= \frac{8 - 3x + 3}{x - 1}
\]
\[
= \frac{11 - 3x}{x - 1}, \quad x \neq 1
\]

Example 3:
The following information refers to the functions \( f \) and \( g \).
\[
f : x \rightarrow 4 - 3x
\]
\[
g : x \rightarrow \frac{3}{x - 2}, \quad x \neq 2
\]
Find
(a) the value of \(gf(1)\)
(b) \(f^2\)

Solution:
For part (a), at first we have to find \(gf(x)\).

Given \(f(x) = 4 - 3x\) and \(g(x) = \frac{3}{x-2}, x \neq 2\)

\[
gf(x) = g[f(x)] = g(4 - 3x) = \frac{3}{(4 - 3x) - 2} = \frac{3}{2 - 3x}
\]

\[
gf(1) = \frac{3}{2 - 3(1)} = \frac{3}{2 - 3} = \frac{3}{-1} = -3
\]

For part (b), \(f^2\) means \(ff\).

Given \(f(x) = 4 - 3x\)

\[
ff(x) = f[f(x)] = f[4 - 3x] = 4 - 3(4 - 3x) = 4 - 12 + 9x = 9x - 8
\]

\(f^2(x) = 9x - 8\)
1.3.1 Determining one of the function in a given composite function

(1) The following information refers to the functions $f$ and $f \circ g$.

$$f : x \rightarrow 2x - 3$$

$$f \circ g : x \rightarrow \frac{11 - 3x}{x - 1}, x \neq 1$$

Find function $g$.

**Solution:**

Given $f(x) = 2x - 3$ and $f \circ g(x) = \frac{11 - 3x}{x - 1}, x \neq 1$

If $f(x) = 2x - 3$,

If $f(y) = 2y - 3$

So $f[g(x)] = 2[g(x)] - 3$

$f \circ g(x) = 2[g(x)] - 3$

Given $f \circ g(x) = \frac{11 - 3x}{x - 1}, x \neq 1$.

Compare the two equations.

Hence,

$$2[g(x)] - 3 = \frac{11 - 3x}{x - 1}$$

$$2[g(x)] = \frac{11 - 3x}{x - 1} + 3$$

$$2[g(x)] = \frac{11 - 3x + 3(x - 1)}{x - 1}$$

$$2[g(x)] = \frac{11 - 3x + 3x - 3}{x - 1}$$

$$2[g(x)] = \frac{8}{x - 1}$$

$$[g(x)] = \frac{8}{2(x - 1)}$$

Remember the concept...

$f(x) = 2x - 3$

$f(k) = 2k - 3$

$f(3) = 2(3) - 3$

$f[g(x)] = 2g(x) - 3$
\[ g(x) = \frac{4}{x-1} \]

Hence \( g : x \rightarrow \frac{4}{x-1}, x \neq 1 \)

(2) The following information refers to the functions \( f \) and \( fg \).

\[
\begin{align*}
  g : x &\rightarrow \frac{4}{x-1} & g : x &\rightarrow \frac{4}{x-1} \\
  fg : x &\rightarrow \frac{11-3x}{x-1}, x \neq 1 & fg : x &\rightarrow \frac{11-3x}{x-1}, x \neq 1
\end{align*}
\]

Find function \( f \).

Solution:

\[
f[g(x)] = \frac{11-3x}{x-1}
\]

\[
f\left( \frac{4}{x-1} \right) = \frac{11-3x}{x-1}
\]

Let

\[
\frac{4}{x-1} = y
\]

1. \( 4 = xy - y \)

2. \( xy = 4 + y \)

3. \( x = \frac{4+y}{y} \)

Substitute 2 and 3 into 1,

\[
f(y) = \frac{11-3\left( \frac{4+y}{y} \right)}{\frac{4+y}{y} - 1}
\]

\[
= \frac{11-\left( \frac{12+3y}{y} \right)}{\frac{4+y}{y} - \frac{y}{y}}
\]

Convert 1 to a fraction
\[ \frac{11y - 12 - 3y}{y} = \frac{4 + y - y}{y} \]
\[ \frac{11y - 12 - 3y}{y} = \frac{4}{y} \]
\[ \frac{8y - 12}{y} = \frac{4}{y} \]
\[ 8y - 12 = 4 \]
\[ \frac{8y - 12}{y} \times y = 4 \]
\[ = 2y - 3 \]
\[ f(y) = 2y - 3 \]
\[ f(x) = 2x - 3 \]

**EXERCISE 1.3**

1. Function \( f \) and \( gf \) are given as \( f(x) = \frac{x + 1}{2} \) and \( gf(x) = 3x + 2 \) respectively. Find function \( g \).

2. Given \( f(x) = 5 - 3x \) and \( g(x) = 4x \). Find function \( fg \).

3. Given \( gf(x) = x - 6 \) and \( g(x) = 4x - 3 \). Find
   (i) function \( f \)
   (ii) function \( fg \)

4. Given the function \( w(x) = 3 - 2x \) and \( v(x) = x^2 - 1 \), find the functions
   (a) \( wv \)
   (b) \( vw \)

5. Given the function \( w : x \rightarrow x + 2 \) and \( v(x) \rightarrow \frac{2x - 1}{x - 1}, x \neq k \),
   (a) state the value of \( k \)
   (b) find
      (i) \( wv(x) \)
      (ii) \( vw(x) \)
1.4 INVERSE FUNCTIONS

If \( f(x) = y \) then \( f^{-1}(y) = x \)

**Example 1**

Given \( f : x \rightarrow 3x + 1 \)

If \( x = 1 \),

\[
  f(1) = 3(1) + 1 = 3 + 1 = 4
\]

Hence,

\( f(1) = 4 \)

So,

\( f^{-1}(4) = 1 \)

let

\[
  3x + 1 = y
\]

\[
  3x = y - 1
\]

\[
  x = \frac{y - 1}{3}
\]

\[
  f^{-1}(y) = \frac{y - 1}{3}
\]

\[
  f^{-1}(x) = \frac{x - 1}{3}
\]

\[
  f^{-1}(4) = \frac{4 - 1}{3} = 1
\]
1.4.1 Determining the inverse function

**Example 1:**

Given \( g: x \to \frac{5}{x-2}, \ x \neq 2 \). Determine the inverse function \( g^{-1} \).

**Solution:**

let
\[
\frac{5}{x-2} = y
\]
\[xy - 2y = 5\]
\[xy = 5 + 2y\]
\[x = \frac{5 + 2y}{y}\]
\[g^{-1}(y) = \frac{5 + 2y}{y}\]
\[g^{-1}(x) = \frac{5 + 2x}{x}, \ x \neq 0\]

**Example 2:**

The function \( f \) is defined by \( f: x \to 3 - 4x \). Find \( f^{-1}(-5) \)

1. **Method 1**

Let
\[3 - 4x = y\]
\[4x = 3 - y\]
\[x = \frac{3 - y}{4}\]
\[f^{-1}(y) = \frac{3 - y}{4}\]
\[f^{-1}(x) = \frac{3 - x}{4}\]
\[f^{-1}(-5) = \frac{3 - (-5)}{4}\]
\[= \frac{8}{4}\]
2. **Method 2**

We know that 
\[ f(x) = 3 - 4x \]

And 
\[ f^{-1}(-5) = y \]

Hence,
\[
\begin{align*}
3 - 4x &= -5 \\
4x &= 3 + 5 \\
4x &= 8 \\
x &= 2
\end{align*}
\]
So,
\[ f^{-1}(-5) = 2 \]

---

**EXERCISE 1.4**

1. Given \( f(x) = 5 - 3x \) and \( g(x) = 4x \). Find
   (i) the value of \( f^{-1}(-4) \)
   (ii) the value of \( (gf)^{-1}(-4) \)

2. Given \( g(x) = 5 - 3x \) and \( gf^{-1}(x) = 4x \). Find function \( f \).

3. Given \( gf(x) = x - 6 \) and \( g(x) = 4x - 3 \). Find
   (i) \( (gf)^{-1} \)
   (ii) the value of \( fg(2) \)

4. Given \( f : x \rightarrow \frac{2x}{x - 1}, x \neq 1 \),
   (a) find the function \( f^{-1} \)
   (b) state the value of \( x \) such that \( f^{-1} \) does not exist.

5. Show that the function \( f : x \rightarrow \frac{x}{x - 1}, x \neq 1 \), has an inverse function that maps onto its self.

6. If \( h : x \rightarrow x - 2 \), find the value of \( p \) given that \( h^{-1}(p) = 8 \).
1.5 ABSOLUTE FUNCTION

Absolute function means the value of image is always positive because of the modulus sign.

Given \( g : x \to 2 - x \) and \( y = |g(x)| \). Find the value of \( y \) if \( x = 3 \)

\[
\begin{align*}
y &= |g(x)| \\
&= |2 - 3| \\
&= 1
\end{align*}
\]

Replace \( g(x) \) with \( 2 - x \)

This is called modulus. Any value in the modulus would be positive. If the number is positive, it remains positive but it is negative it would change to positive when remove the modulus sign.

1.5.1 Graph of absolute function

Example:

Given \( f(x) = 2 - x \) and \( y = |f(x)| \). Sketch the graph of \( y = |f(x)| \) for the domain \( 0 \leq x \leq 6 \). Hence state the corresponding range of \( y \).

\[
\begin{align*}
y &= |f(x)| \\
&= |2 - x|
\end{align*}
\]

Replace \( f(x) \) with its image which is \( 2 - x \)

Minimum value of \( y = 0 \)

When \( 2 - x = 0 \)

\( x = 2 \)  

i) \( x = 0 \)

\[
\begin{align*}
y &= |2 - 0| \\
&= 2
\end{align*}
\]

(ii) \( x = 6 \)

\[
\begin{align*}
y &= |2 - 6| \\
&= 4
\end{align*}
\]

Minimum value of \( y \) must be zero because of absolute value cannot be negative so the minimum it could be is zero.

Then, sketch the graph using the points.
Hence, the corresponding range of $y$ is $0 \leq y \leq 4$

:: Some important information to be understood ::

1. Concept Of Solving Quadratic Equation by factorization

To make a product equal to zero, one of them or both must be equal to zero. We do not know either $x-1$ is equal to zero or $x-4$ equal to zero so that is why we use the word ‘or’ not ‘and.’

Example:

Given $f(x) = 2x^2 - 5x$. Find the possible values of $k$ such that $f^{-1}(3) = k$

Solution:

Hence,

$2x^2 - 5x = 3$

The general form of quadratic equation is $ax^2 + bx + c = 0$. To factorize and use the concept, the equation must be equal to zero.
2x + 1 = 0 or \( x - 3 = 0 \)
\[ x = -\frac{1}{2} \quad \text{or} \quad x = 3 \]

\( f^{-1}(3) = -\frac{1}{2} \) or \( f^{-1}(3) = 3 \)

Given that \( f^{-1}(3) = k \)

\[ k = -\frac{1}{2} \quad \text{or} \quad k = 3 \]

To make a product equal to zero, one of them or both must be equal to zero. We do not know either 2x+1 is equal to zero or x-3 equal to zero so that is why we use the word ‘or’ not ‘and.’

### 2. Concept of Square root and any root which is multiple of 2

In lower secondary we have learned that if \( x^2 = 4 \) then \( x = 2 \). Actually the right answer for it is not 2 but \( \pm 2 \) (\( \pm 2 \) is +2 and -2). This is because if \( (2)^2 \) the answer is 4 and if \( (-2)^2 \), the answer is also 4. So the square root of 4 is \( \pm 2 \). It is also the same for fourth root, sixth root and so on. But in certain situation, the answer will be only one.

**Example:**

Solve the equation \( 4x^2 = 64 \).

**Solution:**

\[ 4x^2 = 64 \]
\[ x^2 = 16 \]
\[ x = \pm 4 \]

### 3. Comparison method

**Example:**

Given the functions \( g : x \rightarrow \frac{h}{x-2} \), \( x \neq 2 \) and \( g^{-1} : x \rightarrow \frac{12 + kx}{x} \), \( x \neq 0 \) where \( h \) and \( k \) are constants, find the value of \( h \) and of \( k \).

**Solution:**

From the information given, we know that the method that we have to use to solve the question is comparison method.
First, we have to find the inverse function of g.

Let
\[
\frac{h}{x - 2} = y
\]
\[yx - 2y = h\]
\[yx = h + 2y\]
\[x = \frac{h + 2y}{y}\]

\(g^{-1}: y \rightarrow \frac{h + 2y}{y}\)
\[g^{-1}: x \rightarrow \frac{h + 2x}{x}, x \neq 0\]  \(\text{1}\)

Given that
\[g^{-1}: x \rightarrow \frac{12 + kx}{x}, x \neq 0\]  \(\text{2}\)

Both of them are inverse function of g.

So compare \(\text{1}\) and \(\text{2}\),
\[g^{-1}: x \rightarrow \frac{(h) + (2)x}{x}, x \neq 0\]
\[g^{-1}: x \rightarrow \frac{(12) + (k)x}{x}, x \neq 0\]

Hence,
\(h = 12\) and \(k = 2\)
CHAPTER REVIEW EXERCISE

1. Diagram 1 shows the relation between set J and set K.

![Diagram 1](image)

**Diagram 1**

**State**

(a) the codomain of the relation,
(b) the type of the relation.

2. Diagram 2 shows the graph of the function $y = f(x)$.

![Diagram 2](image)

It is given that $f(x) = ax + b$.

**Find**

(a) the value of $a$ and of $b$,
(b) the value of $m$ if $m$ is mapped onto itself under the function $f$,
(c) the value of $x$ if $f^{-1}(x) = f^2(x)$.

3. Function $h$ is given as $h(x) = \frac{Kx - 4}{x - 1}, x \neq k$. Find the value of $k$.

4. Given function $g(x) = hx + 5$ and $g^{-1}(x) = \frac{x + k}{2}$, find the values of $h$ and $k$.

5. Given $g : x \rightarrow \frac{3}{x - 2}, x \neq 2$. Find the values of $x$ if $g(x) = x$. 
6. Functions \( f \) and \( g \) are defined by \( f : x \to 3x - h \) and \( g : x \to \frac{k}{x}, x \neq 0 \), where \( h \) and \( k \) are constants. It is given that \( f^{-1}(10) = 4 \) and \( fg(2) = 16 \). Find the value of \( h \) and of \( k \).

7. Given that \( f(x) = 3x - 2 \) and \( fg(x) = x^2 - 1 \), find
   (i) the function \( g \)
   (ii) the value of \( gf(3) \)

8. Given \( f(x) = 2x + 2 \) and \( y = |f(x)| \). Sketch the graph of \( y = |f(x)| \) for the domain \( 0 \leq x \leq 4 \). Hence state the corresponding range of \( y \).

9. Given that function \( h \) is defined as \( h : x \to px + 12 \). Given that the value of 3 maps onto itself under the function \( h \), find
   (a) the value of \( p \)
   (b) the value of \( h^{-1}(-2) \)

10. Given that the functions \( h(x) = 2x - 1 \) and \( hh(x) = px + q \), where \( p \) and \( q \) are constants, calculate
    (a) the value of \( p \) and \( q \)
    (b) the value of \( m \) for which \( h(-1) = 2m \)

11. Function \( g \) and composite function \( fg \) are defined by \( g : x \to x + 3 \) and \( fg : x \to x^2 + 6x + 7 \).
    (a) Sketch the graph of \( y = |g(x)| \) for the domain \( -5 \leq x \leq 3 \).
    (b) Find \( f(x) \).

12. Given quadratic function \( g(x) = 2x^2 - 3x + 2 \). find
    (a) \( g(3) \),
    (b) the possible values of \( x \) if \( g(x) = 22 \)

13. Functions \( f \) and \( g \) are given as \( f : x \to x^2 \) and \( g : x \to px + q \), where \( p \) and \( q \) are constants.
    (a) Given that \( f(1) = g(1) \) and \( f(3) = g(5) \), find the value of \( p \) and of \( q \).
    (b) By using the values of \( p \) and \( q \) obtained in (a), find the functions
        (i) \( gg \)
        (ii) \( g^{-1} \)

14. If \( w(x) = 3x - 6 \) and \( v(x) = 6x - 1 \), find
    (a) \( wv(x) \)
    (b) the value of \( x \) such that \( wv(-2x) = x \).

15. Given \( f(x) = |3x - 6| \). Find the values of \( x \) if \( f(x) = 3 \).